Problem 1.10

Vectors (and scalars, and tensors) are defined by their behavior under a *rotation* of the coordinate axes. But it is sometimes of interest to enquire how they might transform (using displacements as a model) under other coordinate changes – translations, say, or inversions.

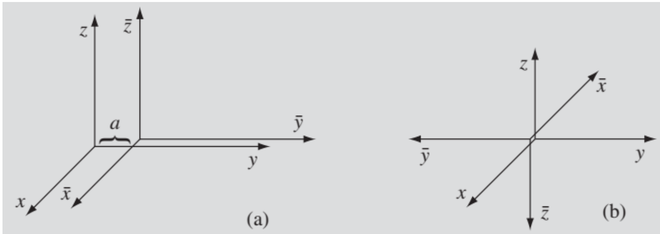


Fig. 1.16

(a) How do the components of a vector[5] transform under a **translation** of coordinates ($\bar{x} = x$, $\bar{y} = y - a$, $\bar{z} = z$, Fig. 1.16a)?

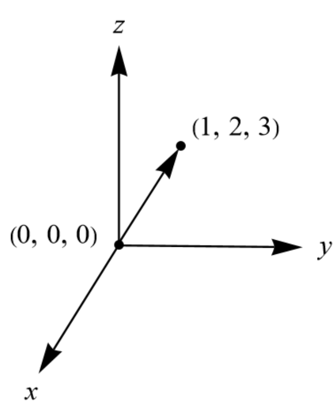
(b) How do the components of a displacement transform under an **inversion** of coordinates ($\bar{x} = -x$, $\bar{y} = -y$, $\bar{z} = -z$, Fig. 1.16b)?

(c) How do the components of a cross product (Eq. 1.13) transform under inversion? (The cross product of two vectors is called a **pseudovector** because of this "anomalous" behavior under inversions. It's still a *vector* – that is determined by its behavior under *rotations*; where the distinction is at issue I'll call a vector with the "normal" behavior under inversions an **ordinary vector**.) Is the cross product of two pseudovectors an ordinary vector, or a pseudovector? Name two pseudovector quantities in classical mechanics.

(d) How does the scalar triple product of three vectors transform under inversions? (Such an object is called a **pseudoscalar.**)

Solution

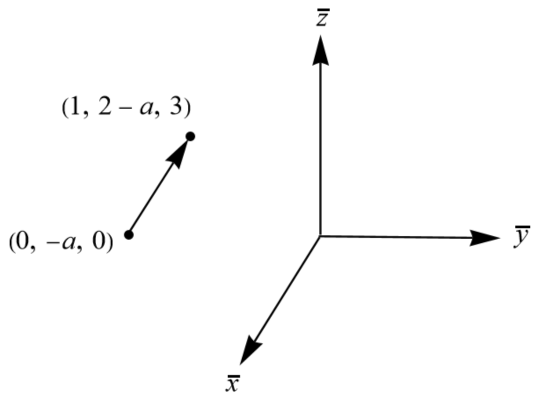
Consider the displacement vector from $(0, 0, 0)$ to $(1, 2, 3)$ in an xyz-coordinate system.



[5] *Beware:* The vector $\mathbf{r}$ (Eq. 1.19) goes from a specific point in space (the origin, $\mathscr{O}$) to the point $P = (x, y, z)$. Under a translation the *new* origin ($\bar{\mathscr{O}}$) is at a different location, and the arrow from $\bar{\mathscr{O}}$ to P is a completely different vector. The original vector $\mathbf{r}$ still goes from $\mathscr{O}$ to P, regardless of the coordinates used to label these points.

Part (a)

If the coordinate system gets translated ($\bar{x} = x$, $\bar{y} = y - a$, $\bar{z} = z$), the coordinates of the tail and head of the vector change.



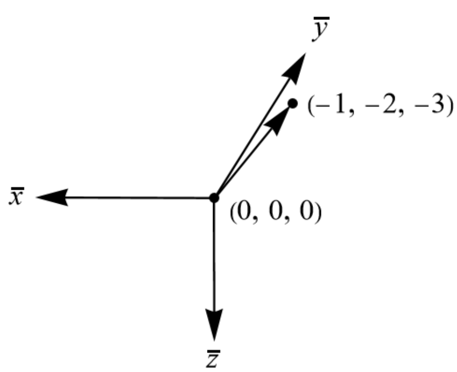
The vector in this new coordinate system is

$$\begin{aligned}\bar{\mathbf{r}} &= \langle 1, 2-a, 3\rangle - \langle 0, -a, 0\rangle \\ &= \langle 1, 2, 3\rangle \\ &= \mathbf{r}.\end{aligned}$$

Therefore, the components of a vector do not change under a translation of coordinates.

Part (b)

If the coordinate system gets inverted ($\bar{x} = -x$, $\bar{y} = -y$, $\bar{z} = -z$), the coordinates of the tail and head of the vector change.



The vector in this new coordinate system is

$$\begin{aligned}\bar{\mathbf{r}} &= \langle -1, -2, -3\rangle - \langle 0, 0, 0\rangle \\ &= \langle -1, -2, -3\rangle \\ &= -\langle 1, 2, 3\rangle = -\mathbf{r}.\end{aligned}$$

Therefore, the components of a vector become negative under an inversion of coordinates.

Part (c)

The cross product of two vectors, **A** and **B**, is

$$\begin{aligned}
\mathbf{A} \times \mathbf{B} &= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i A_i\right) \times \left(\sum_{j=1}^{3} \boldsymbol{\delta}_j B_j\right) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3} (\boldsymbol{\delta}_i \times \boldsymbol{\delta}_j) A_i B_j \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_k \varepsilon_{ijk} A_i B_j.
\end{aligned}$$

Under an inversion of coordinates, the components of each vector become negative.

$$\begin{aligned}
\overline{\mathbf{A} \times \mathbf{B}} &= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_k \varepsilon_{ijk} (-A_i)(-B_j) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_k \varepsilon_{ijk} A_i B_j \\
&= \mathbf{A} \times \mathbf{B}
\end{aligned}$$

Therefore, the components of a cross product do not change under an inversion of coordinates, which is unlike the behavior of a normal vector. The cross product of two cross products is

$$\begin{aligned}
(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D}) &= \left[\left(\sum_{i=1}^{3} \boldsymbol{\delta}_i A_i\right) \times \left(\sum_{j=1}^{3} \boldsymbol{\delta}_j B_j\right)\right] \times \left[\left(\sum_{k=1}^{3} \boldsymbol{\delta}_k C_k\right) \times \left(\sum_{l=1}^{3} \boldsymbol{\delta}_l D_l\right)\right] \\
&= \left[\sum_{i=1}^{3}\sum_{j=1}^{3} (\boldsymbol{\delta}_i \times \boldsymbol{\delta}_j) A_i B_j\right] \times \left[\sum_{k=1}^{3}\sum_{l=1}^{3} (\boldsymbol{\delta}_k \times \boldsymbol{\delta}_l) C_k D_l\right] \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} [(\boldsymbol{\delta}_i \times \boldsymbol{\delta}_j) \times (\boldsymbol{\delta}_k \times \boldsymbol{\delta}_l)] A_i B_j C_k D_l.
\end{aligned}$$

Under an inversion of coordinates, the components of each vector become negative.

$$\begin{aligned}
\overline{(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D})} &= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} [(\boldsymbol{\delta}_i \times \boldsymbol{\delta}_j) \times (\boldsymbol{\delta}_k \times \boldsymbol{\delta}_l)] (-A_i)(-B_j)(-C_k)(-D_l) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} [(\boldsymbol{\delta}_i \times \boldsymbol{\delta}_j) \times (\boldsymbol{\delta}_k \times \boldsymbol{\delta}_l)] A_i B_j C_k D_l \\
&= (\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D})
\end{aligned}$$

The cross product of two pseudovectors is a pseudovector as well. Two quantities in classical mechanics that are cross products are angular momentum, $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, and torque, $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$.

Part (d)

The scalar triple product of three vectors is

$$\begin{aligned}
\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) &= \left(\sum_{i=1}^{3}\boldsymbol{\delta}_i A_i\right)\cdot\left[\left(\sum_{j=1}^{3}\boldsymbol{\delta}_j B_j\right)\times\left(\sum_{k=1}^{3}\boldsymbol{\delta}_k C_k\right)\right] \\
&= \left(\sum_{i=1}^{3}\boldsymbol{\delta}_i A_i\right)\cdot\left[\sum_{j=1}^{3}\sum_{k=1}^{3}(\boldsymbol{\delta}_j\times\boldsymbol{\delta}_k)B_jC_k\right] \\
&= \left(\sum_{i=1}^{3}\boldsymbol{\delta}_i A_i\right)\cdot\left(\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\boldsymbol{\delta}_l\varepsilon_{jkl}B_jC_k\right) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}(\boldsymbol{\delta}_i\cdot\boldsymbol{\delta}_l)\varepsilon_{jkl}A_iB_jC_k.
\end{aligned}$$

Under an inversion of coordinates, the components of each vector become negative.

$$\begin{aligned}
\overline{\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})} &= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}(\boldsymbol{\delta}_i\cdot\boldsymbol{\delta}_l)\varepsilon_{jkl}(-A_i)(-B_j)(-C_k) \\
&= -\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}(\boldsymbol{\delta}_i\cdot\boldsymbol{\delta}_l)\varepsilon_{jkl}A_iB_jC_k \\
&= -\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})
\end{aligned}$$

The scalar triple product becomes negative under an inversion of coordinates, which is unlike the behavior of a normal scalar.